

Reflection Matrices for Integrable $N = 1$ Supersymmetric Theories

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Abstract

We study two-dimensional integrable $N = 1$ supersymmetric theories (without topological charges) in the presence of a boundary. We find a universal ratio between the reflection amplitudes for particles that are related by supersymmetry and we propose exact reflection matrices for the supersymmetric extensions of the multi-component Yang-Lee models and for the breather multiplets of the supersymmetric sine-Gordon theory. We point out the connection between our reflection matrices and the classical boundary actions for the supersymmetric sine-Gordon theory as constructed by Inami, Otake and Zhang [1].

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I. INTRODUCTION

Quantum field theory (QFT) has been very successful in the description of physical phenomena in a wide area, ranging from particle physics at the one end to statistical mechanics and condensed matter physics at the other. When working out applications of QFT to physical systems, one typically finds that, while realistic theories tend to be hard to analyze, the ones that we can solve in closed form tend to be not so realistic. However, a number of successes (in particular some recent ones, where exactly solvable QFT's were applied to some two dimensional condensed matter systems) have clearly shown that the situation is not so bad after all, and that there is a non-empty overlap between 'realistic' and 'solvable' QFT's. Especially promising in this respect are the integrable QFT's in two dimensions. Due to the existence of an infinite number of commuting conserved charges, these are highly constrained theories that display some very nice features, in particular the absence of particle production in scattering processes and the factorizability of the S -matrix.

It is interesting to consider QFT's which, apart from being integrable, are at the same time supersymmetric. In many examples [2,3], the $N = 1$ or $N = 2$ supersymmetry algebra contains topological charges, and the fundamental particles should be viewed as kinks in a non-trivial potential. It is possible, however, to have $N = 1$ supersymmetric theories without topological charges [4–6], and it is on this case that we shall focus in this paper. One motivation for including supersymmetry in integrable theories comes from particle physics and the superstring connection. In addition, it may be observed that supersymmetric factorizable S -matrices are among the simplest ones that are non-diagonal. Certain procedures that are problematic for non-diagonal S -matrices, such as for example the Thermodynamic Bethe Ansatz (TBA), become manageable when the non-diagonal S -matrices are controlled by an underlying symmetry such as supersymmetry [7–9]. This point was made very clearly in [10,8,9], where it was shown that $N = 1$ supersymmetry leads to a 'free fermion condition' for certain two-particle S -matrices, and that this free fermion condition makes it possible to perform the TBA analysis in closed form.

In [4–6] the exact S -matrices were found for the $N = 1$ supersymmetric Yang-Lee model and its multicomponent generalizations, and for the bound state multiplets of the $N = 1$ supersymmetric (susy) sine-Gordon theory. In all these examples, the $N = 1$ supersymmetry algebra is free of topological charges. The multicomponent supersymmetric Yang-Lee models are supersymmetric versions of the models studied by Freund, Klassen and Melzer (FKM) in [11]. The latter correspond to the minimal reductions of the $A_{2n}^{(2)}$ affine Toda models. We will call their supersymmetric extensions the “susy FKM” models.

The natural next step after the study of exact bulk S -matrices is to study the same systems in the presence of a boundary. Again the motivation comes from (open) string theory and the study of statistical mechanics and condensed matter systems with non-trivial behavior at a boundary. Examples of the latter are the various Kondo systems and edge current dynamics in the quantum Hall effect.

In this paper we return to the supersymmetric S -matrices studied in [4–6,9] and study the introduction of a boundary in those theories. Our strategy will be to assume boundary conditions (interactions) that preserve both the integrability and a combination of the left and right supersymmetries. At the quantum level, these properties imply conditions which will allow us to determine the exact reflection matrices. We can then compare those with predictions based on a purely classical analysis of supersymmetric boundary conditions, see, e.g., [1].

This paper is organized as follows. In section II we discuss the supersymmetric bulk theories (including their exact S -matrices) that we are going to “boundarize”, with some special emphasis on the bound state structure. Supersymmetric reflection matrices are introduced in section III. We study some of their general features and consider the possibility of boundary bound states. In section IV we discuss the bosonic reflection matrices for the FKM models and for the bound states of the sine-Gordon model. Complete supersymmetric reflection matrices are presented in section V (for the susy FKM models) and VI (for the susy sine-Gordon model). In VI, we also discuss the relation with the work of [1]. In the final section VII we discuss possible extensions and we present our conclusions.

II. S -MATRICES WITH $N = 1$ SUPERSYMMETRY

In this section we introduce a “universal” bose-fermi S -matrix $S_{BF}(\theta)$, which is relevant for theories with $N = 1$ supersymmetry without topological charges. We discuss specific examples, which are the susy FKM series and the susy sine-Gordon theory. As a warm-up, we first review some of the basic facts about particle scattering in integrable supersymmetric theories in $1 + 1$ dimensions. Many of the facts and features discussed here are shared by general integrable field theories, but to keep the discussion short we will introduce those concepts directly in the case of supersymmetric models. For more details see [4–6].

Asymptotic States and the S -matrix

The particles in a massive supersymmetric theory are arranged in supermultiplets, each having one boson and one fermion (b_i, f_i) of equal mass m_i , $i = 1, \dots, n$. The particle operators $A_a(\theta)$ (for bosons or fermions¹) depend on the rapidity variable θ , which is such that the on-shell energy and momentum are given by $p_0 = m_a \cosh(\theta)$ and $p_1 = m_a \sinh(\theta)$, respectively. The in (out) N -particle states are written as

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2) \dots A_{a_N}(\theta_N)\rangle_{in(out)} , \quad (2.1)$$

where $\theta_1 \geq \theta_2 \geq \dots \geq \theta_N$ for in-states and the other way around for out-states.

Since we are dealing with integrable field theories, we can reduce any multiparticle scattering process to two-body scattering processes. For this reason, the two-body S -matrix is a key-quantity for understanding the structure of the theory. We define it by

$$S_{a_1 a_2}^{b_1 b_2}(\theta_1 - \theta_2) |A_{b_2}(\theta_2)A_{b_1}(\theta_1)\rangle_{out} = |A_{a_1}(\theta_1)A_{a_2}(\theta_2)\rangle_{in} . \quad (2.2)$$

See figure 1. Notice the inversion of the rapidities for the in and out states.

¹We shall reserve letters a, b, a_1, \dots for particles b_i or f_i and i, j, i_1, \dots for the multiplet index $i = 1, 2, \dots, n$.

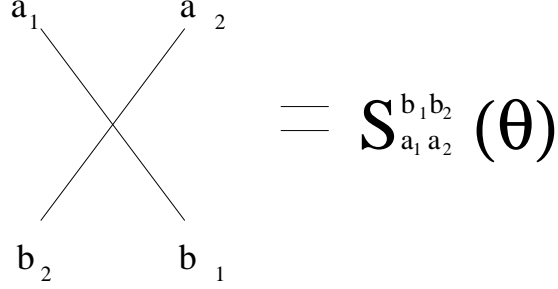


FIG.1 The two-body S -matrix element

In the next subsection we discuss the supersymmetry algebra and the action of the supercharges on the asymptotic states.

The Supersymmetry Algebra

The supersymmetry of a $d = 1 + 1$ particle theory is expressed by the presence of conserved supercharges $Q_+(\theta)$ and $Q_-(\theta)$ that satisfy

$$\begin{aligned} \{Q_L, Q_\pm\} &= 0 \\ Q_+^2 &= p_0 + p_1, \quad Q_-^2 = p_0 - p_1 \\ \{Q_+, Q_-\} &= 0. \end{aligned} \tag{2.3}$$

The Q_L operator measures the fermion number of asymptotic states. We could have chosen the anti-commutator $\{Q_+, Q_-\}$ to be a non-zero c-number Z . This would correspond to having a topological charge. We shall here restrict our attention to a realization of the superalgebra that has $Z = 0$.

The supersymmetry charges $Q_\pm(\theta)$ act on asymptotic one-particle states according to

$$\begin{aligned} Q_+(\theta) |b(\theta)\rangle &= \sqrt{m} e^{\frac{\theta}{2}} |f(\theta)\rangle, & Q_+(\theta) |f(\theta)\rangle &= \sqrt{m} e^{\frac{\theta}{2}} |b(\theta)\rangle \\ Q_-(\theta) |b(\theta)\rangle &= i\sqrt{m} e^{-\frac{\theta}{2}} |f(\theta)\rangle, & Q_-(\theta) |f(\theta)\rangle &= -i\sqrt{m} e^{-\frac{\theta}{2}} |b(\theta)\rangle. \end{aligned} \tag{2.4}$$

This corresponds to the following realization

$$Q_+(\theta) = \sqrt{m} e^{\frac{\theta}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_-(\theta) = \sqrt{m} e^{-\frac{\theta}{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Q_L(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

We have to define the action of the supercharges on multi-particle states. It is easy to check that the following expression works

$$\mathcal{Q}_+(\theta) = \sum_{l=1}^N \mathcal{Q}_{+l}(\theta), \quad \mathcal{Q}_-(\theta) = \sum_{l=1}^N \mathcal{Q}_{-l}(\theta), \quad (2.6)$$

where $\mathcal{Q}_{\pm l}(\theta)$ is defined by

$$\begin{aligned} \mathcal{Q}_{\pm l}(\theta) |A_{a_1}(\theta_1) \dots A_{a_N}(\theta_N)\rangle = \\ \prod_{k=1}^{l-1} (-1)^{F_{a_k}} |A_{a_1}(\theta_1) \dots A_{a_{l-1}}(\theta_{l-1}) (A\mathcal{Q}_{\pm})_{a_l}(\theta_l) A_{a_{l+1}}(\theta_{l+1}) \dots A_{a_N}(\theta_N)\rangle, \end{aligned} \quad (2.7)$$

where $(-1)^F$ is +1 for a boson and -1 for a fermion.

Given this brief description of the supersymmetry algebra and how the supercharges act on the multi-particle asymptotic states we can move on and discuss exact S -matrices with $N = 1$ supersymmetry. We should stress again that we have chosen the specific realization (2.5) (with zero topological charge) and that this by no means exhausts the possible $N = 1$ supersymmetric theories.

The Structure of the S -matrix

We concentrate on supersymmetric two-particle S -matrices that can be written in the following factorized form

$$S = S_B \otimes S_{BF}, \quad (2.8)$$

where S_B is the S -matrix of the bosonic sector (we shall be thinking of a diagonal bosonic S -matrix) and S_{BF} is the supersymmetric piece, responsible for mixing bosons and fermions. Imposing that this S -matrix commutes with the supersymmetry charges $\mathcal{Q}_{\pm}(\theta)$, one finds that $S_{BF}(\theta)$ gets fixed up to one unknown function. Explicitly, one finds the following scattering matrix for the particles in the i -th and j -th supermultiplets [5]

$$S_{BF}^{[ij]}(\theta) = f^{[ij]}(\theta) \begin{pmatrix} 1 - t\tilde{t} & 0 & 0 & -i(t + \tilde{t}) \\ 0 & -t + \tilde{t} & 1 + t\tilde{t} & 0 \\ 0 & 1 + t\tilde{t} & t - \tilde{t} & 0 \\ -i(t + \tilde{t}) & 0 & 0 & 1 - t\tilde{t} \end{pmatrix} + g^{[ij]}(\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (2.9)$$

where $t = \tanh([\theta + \log(m_i/m_j)]/4)$ and $\tilde{t} = \tanh([\theta - \log(m_i/m_j)]/4)$ and where we have absorbed a factor of $e^{-\frac{\pi i}{4}}$ in the definition of the basic fermionic states $|f_i\rangle$. The two functions $f^{[ij]}(\theta)$ and $g^{[ij]}(\theta)$ are related and have to be fixed by the other conditions that the S -matrix satisfies. The Yang-Baxter equation fixes the ratio of $f^{[ij]}(\theta)$ and $g^{[ij]}(\theta)$ up to a single constant α

$$f^{[ij]}(\theta) = \frac{\alpha}{4i} \sqrt{m_i m_j} \left[\frac{2 \cosh(\theta/2) + (\rho^2 + \rho^{-2})}{\cosh(\theta/2) \sinh(\theta/2)} \right] g^{[ij]}(\theta) , \quad (2.10)$$

where $\rho = (m_i/m_j)^{1/4}$.

We have to use analyticity, crossing symmetry and unitarity to fix the function $g^{[ij]}(\theta)$. One expression for $g^{[ij]}(\theta)$ is

$$g^{[ij]}(\theta) = \frac{g_{\Delta_1}(\theta) g_{\Delta_2}(\theta)}{g_{\Delta_3}(\theta)} , \quad (2.11)$$

where

$$g_{\Delta}(\theta) = \frac{\sinh(\frac{\theta}{2})}{\sinh(\frac{\theta}{2}) + i \sin(\Delta\pi)} \exp \left(i \int_0^\infty \frac{dt \sinh(\Delta t) \sinh((1-\Delta)t)}{t \cosh^2(\frac{t}{2}) \cosh(t)} \sin(\frac{t\theta}{\pi}) \right) \quad (2.12)$$

and $\Delta_1 = \frac{1}{2}(i+j)\beta$, $\Delta_2 = \frac{1}{2}(1-(i-j)\beta)$ and $\Delta_3 = \frac{1}{2}$. We refer to our previous paper [9] for more details.

We remark that there is an important difference between the periodicity properties of diagonal and non-diagonal S -matrices. In the diagonal case, the conditions for unitarity and crossing symmetry read

$$S_{ij}(\theta) S_{ik}(-\theta) = \delta_{ik} , \quad S_{ij}(\theta) = S_{ij}(i\pi - \theta) , \quad (2.13)$$

and these conditions alone imply immediately that $S_{ij}(\theta)$ is $2\pi i$ -periodic. The same is not true for non-diagonal S -matrices. We shall later see that similar remarks apply to the boundary reflection matrices.

Bound State Structure and Three Point Couplings

Once we have identified a candidate for an exact S -matrix we should study the bound state structure of the theory and check the validity of the bootstrap principle: the particles

that correspond to bound state poles in the S -matrix should be included in the list of asymptotic particles. This then leads to a number of consistency requirements that have to be satisfied by the residues at the bound state poles of the S -matrix. For the supersymmetric theories described here one can derive the following [5].

We denote the 3-point coupling of particles a , b and c by f_{ab}^c . It can be shown that if f_{ij}^k is a non-vanishing coupling of the bosonic sector (described by $S_B^{[ij]}(\theta)$) then the full supersymmetric theory has non-zero couplings $f_{b_ib_j}^{b_k}$, $f_{b_if_j}^{f_k}$, $f_{f_ib_j}^{f_k}$ and $f_{f_if_j}^{b_k}$, satisfying

$$\frac{f_{f_if_j}^{b_k}}{f_{b_ib_j}^{b_k}} = \left(\frac{m_i + m_j - m_k}{m_i + m_j + m_k} \right)^{\frac{1}{2}}. \quad (2.14)$$

Furthermore, the value of the free parameter α in the bose-fermi S -matrix (see (2.10)) can be expressed in terms of the masses m_i , m_j and m_k of any three multiplets i, j, k with non-zero 3-point coupling f_{ij}^k , according to

$$\alpha = -\frac{(2m_i^2m_j^2 + 2m_i^2m_k^2 + 2m_j^2m_k^2 - m_i^4 - m_j^4 - m_k^4)^{\frac{1}{2}}}{2m_im_jm_k}. \quad (2.15)$$

Note that one free constant α has to fit all non-zero couplings of the bosonic theory. Clearly, this will only be possible in very special cases. This then shows the type of supersymmetrization that is described here is only possible for a selected set of bosonic theories.

If the multiplets i , j and k have a non-vanishing coupling, the two-body S -matrix $S^{[ij]}(\theta)$ will have a pole at $\theta = iu_{ij}^k$ where u_{ij}^k satisfies

$$\cos(u_{ij}^k) = \frac{m_k^2 - m_i^2 - m_j^2}{2m_im_j}. \quad (2.16)$$

Clearly, the $u_{ij}^k, u_{jk}^i, u_{ki}^j$ are the external angles of a triangle with sides m_i, m_j and m_k (and so we define the internal angles $\bar{u}_{ij}^k = \pi - u_{ij}^k$).

Examples: Supersymmetric FKM Series and Supersymmetric sine-Gordon

It was observed in [5] that the conditions (2.15) are all satisfied in theories with a mass spectrum

$$m_j = \frac{\sin(j\beta\pi)}{\sin(\beta\pi)}, \quad j = 1, 2, \dots \quad (2.17)$$

and the following allowed fusion pattern: f_{ij}^k is non-vanishing if $k = |i - j|$ or $k = i + j$. In this case $\alpha = -\sin(\beta\pi)$.

This combination of spectrum and fusion rules is known to be realized in at least two (series of) examples: the sine-Gordon theory and the multicomponent Yang-Lee (or FKM [11]) minimal models. In the sine-Gordon case, the above are the masses and fusion rules of the breathers (bound-state) and β is related to the (adjustable) coupling constant β_{sG} . The FKM minimal models correspond to $\beta_{sG} = 1/(2n + 1)$, $n = 1, 2, \dots$. They can to some extent be viewed as consistent truncations of the sine-Gordon bound-state sectors. In the FKM models there are additional non-zero couplings f_{ij}^k for $i + j + k = 2n + 1$ (again, these are consistent with the conditions (2.15)).

One thus expects that the scattering matrices for the $N = 1$ supersymmetric extensions of both the FKM (susy FKM) and the sine-Gordon (susy sine-Gordon) theories will be of the form described above. This has been confirmed in [5,8,9,4,6]. In the remainder of this section we briefly summarize these results.

The susy FKM models are integrable deformations of specific non-unitary superconformal minimal models, labeled by an integer $n = 1, 2, \dots$, and of central charge given by $c_n = -3n(4n + 3)/(2n + 2)$. They are formally defined through a perturbed superconformal field theory with action

$$S_\lambda = S + \lambda \int \overline{G}_{-\frac{1}{2}} G_{-\frac{1}{2}} \phi_{h,h} d^2x , \quad (2.18)$$

where $\phi_{h,h}$ is a primary field in the Neveu-Schwarz sector of the left and right chiral superconformal algebras and $h = h_{(1,3)}$. This is an integrable perturbation that is manifestly supersymmetric. Having introduced a relevant parameter λ , we obtain a massive theory, with particles (b_j, f_j) , $j = 1, 2, \dots, n$, with masses as in (2.17) with $\beta = 1/(2n + 1)$. The S -matrix for these theories takes the form (2.8) with S_B given by the S -matrix of the bosonic multi-component Yang-Lee model as found by Freund, Klassen and Melzer (FKM) in [11]

$$S_B^{[ij]}(\theta) = F_{|i-j|\beta}(\theta) \left[F_{(|i-j|+2)\beta}(\theta) \dots F_{(i+j-2)\beta}(\theta) \right]^2 F_{(i+j)\beta}(\theta) , \quad (2.19)$$

with $F_\alpha(\theta) = \frac{\sinh(\theta) + i \sin(\alpha\pi)}{\sinh(\theta) - i \sin(\alpha\pi)}$ and $\alpha = -\sin(\pi/(2n+1))$. These bosonic S -matrices can be viewed as ‘minimal reductions’ of the S -matrices of the twisted affine Toda theories based on $A_{2n}^{(2)}$.

The first model ($n = 1$) in the susy FKM series is a supersymmetric version of the perturbed Yang-Lee conformal field theory. It has a single massive supermultiplet (b, f) with non-zero self-couplings $f_{bb}^b = \sqrt{3} f_{ff}^b$ that correspond to a pole at $\theta = i\frac{2\pi}{3}$ in the bosonic factor $S_B(\theta)$ of the S -matrix. This theory may justifiably be called the “world’s simplest interesting supersymmetric scattering theory”, and as such it serves as a prototype for theoretical investigations of more complicated supersymmetric models.

The supersymmetric sine-Gordon model is defined by the following action in Euclidean space-time

$$S_{ssG} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}(\partial_y \phi)^2 - \bar{\psi}(\partial_x - i\partial_y)\bar{\psi} + \psi(\partial_x + i\partial_y)\psi - \right. \\ \left. - \frac{m^2}{\beta_{ssG}^2} \cos(\beta_{ssG}\phi) - 2m\bar{\psi}\psi \cos\left(\frac{\beta_{ssG}\phi}{2}\right) \right\}, \quad (2.20)$$

where ϕ is the bosonic field and ψ and $\bar{\psi}$ are the components of a Majorana fermion. The spectrum of the full quantum theory contains (anti-)soliton multiplets and bound state multiplets (b_j, f_j) , $j = 1, 2, \dots < \lambda$, $\lambda = 2\pi(1 - (\beta_{ssG}^2/4\pi))/\beta_{ssG}^2$, of masses (2.17) with $\beta = 1/(2\lambda)$. The S -matrix for the bound state multiplets [4,6] again takes the factorized form (2.8), where the $S_B(\theta)$ are now the breather S -matrices of the bosonic sine-Gordon model. They can be found in the original paper [12].

The truncation from susy sine-Gordon to a minimal susy FKM model is possible when λ is of the form $\lambda = \frac{2n+1}{2}$. Consider for example $n = 1$, which gives $\lambda = \frac{3}{2}$. At this coupling, only the lightest breather multiplet ($j = 1$) is stable. However, it so happens that the second breather multiplet, which occurs as a virtual bound state in the S -matrix of the first multiplet ($f_{11}^2 \neq 0$), has the same mass as the first one. It is then consistent to consider the minimal reduction, where the $j = 1, 2$ bound state multiplets are ‘folded’ onto a single particle with self-coupling ($f_{11}^1 \neq 0$), and this folded theory is essentially the same

as the Yang-Lee scattering theory. This formal connection explains the similarity between the S -matrices for the susy sine-Gordon and susy FKM theories.

III. SUPERSYMMETRIC REFLECTION MATRICES: GENERALITIES

Once we have understood the bulk S -matrix for a physical theory it is natural to study the same theory in the presence of a boundary. The reasons for studying boundary theories are both theoretical and physical. From the theoretical point of view it is an interesting (and difficult) problem to find and identify the boundary reflection matrices that correspond to given boundary conditions or a given boundary action. From the practical point of view, many physical systems such as the Kondo system and edge excitations in the fractional quantum Hall effect correspond to boundary problems.

We should remark that conserved quantities in the bulk may be no longer be conserved after the introduction of a boundary, as is for example the case for linear momentum. So we expect that not all boundary actions lead to integrable boundary theories and if we insist on integrability we shall have to be careful about which boundary action we are picking. In this paper we shall focus on boundary conditions that preserve both the integrability and the supersymmetry of the bulk theory.

In order to have the complete description of a particle theory in the presence of a boundary we have to understand how particles scatter off that boundary. In an integrable theory one expects that this scattering is one-to-one. The corresponding amplitudes are contained in a reflection matrix $R_a^b(\theta)$. This amplitude is shown in figure 2. In this section we review some general features of such reflection matrices and discuss the implementation of supersymmetry. Our presentation follows the spirit and the notations of [13]. We refer to [14] for a study of $N = 2$ supersymmetry in integrable models with a boundary.

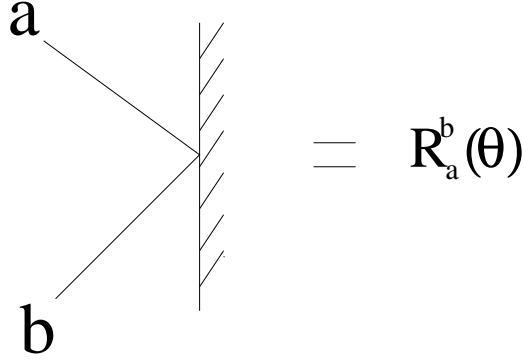


FIG.2 The reflection matrix

We will restrict our discussion to the case where the boundary has no structure, so that the reflection matrix can be assumed to be diagonal, $R_a^b(\theta) = \delta_a^b R_a(\theta)$, no sum over a . (In the presence of boundary bound states more general reflection matrices should be considered, see below.) The one-particle reflection amplitudes are subject to a number of conditions that are most easily understood by drawing some pictures, see [15,13]. In an analogous way as for the bulk theories, we have boundary Yang-Baxter equations (BYBE) (see figure 3) ²,

$$R_{a_2}(\theta_2) S_{a_1 a_2}^{c_1 d_2}(\theta_1 + \theta_2) R_{c_1}(\theta_1) S_{d_2 c_1}^{b_2 b_1}(\theta_1 - \theta_2) = S_{a_1 a_2}^{c_1 c_2}(\theta_1 - \theta_2) R_{c_1}(\theta_1) S_{c_2 c_1}^{b_2 b_1}(\theta_1 + \theta_2) R_{b_2}(\theta_2) . \quad (3.1)$$

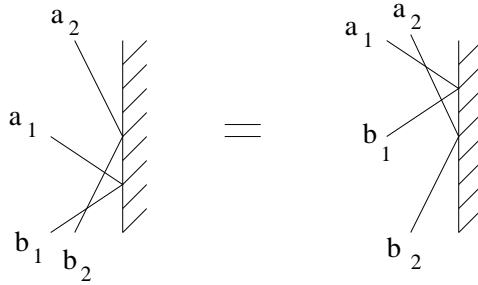


FIG.3 The boundary Yang-Baxter equation

We have also a unitarity condition for the reflection matrix,

$$R_a(\theta) R_a(-\theta) = 1 , \quad (3.2)$$

²no sum over a_1, a_2, b_1, b_2

and a condition called boundary crossing-unitarity

$$R_a(i\frac{\pi}{2} - \theta) = S_{bb}^{aa}(\theta) R_b(i\frac{\pi}{2} + \theta) , \quad (3.3)$$

where we sum over b . If the bulk S -matrix has non-trivial bound state poles there are additional “boundary bootstrap” conditions, which include the identity ³

$$f_c^{ab} R_c(\theta) = f_c^{a_1 b_1} R_b(\theta - i\bar{u}_{bc}^a) S_{b_1 a_1}^{ba}(2\theta + i\bar{u}_{ac}^b - i\bar{u}_{bc}^a) R_{a_1}(\theta + i\bar{u}_{ac}^b) , \quad (3.4)$$

where f_c^{ab} are the fusion constants discussed before and we used the fact that we are dealing with neutral particles, $\bar{a} = a$. See figure 4.

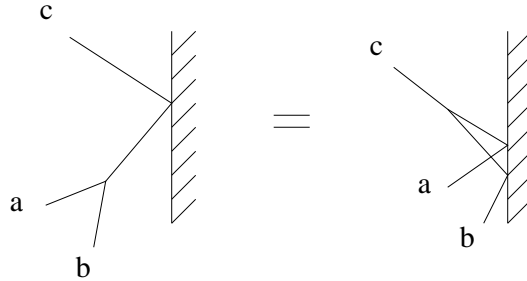


FIG.4 The boundary bootstrap condition

The usual strategy for handling these equations, given the knowledge of a bulk S -matrix, is to first solve the BYBE and then take care of the other equations and of the boundary bootstrap requirements. However, in our case we have supersymmetry as an additional ingredient which we can use to simplify the analysis.

We shall assume that the reflection matrix has a factorized structure similar to that of the bulk S -matrix, namely

$$R(\theta) = R_B(\theta) \otimes R_{BF}(\theta) , \quad (3.5)$$

where $R_B(\theta)$ is the reflection matrix for the bosonic part of the theory, and $R_{BF}(\theta)$ is the “supersymmetric” part of the reflection matrix. Given this form for the reflection matrix,

³summing over a_1 and b_1

all bosonic equations will factor out and we have therefore two problems to consider. The first is to find the reflection matrices $R_B(\theta)$ corresponding to the bosonic sector. In the cases at hand these will be reflection matrices for the FKM series (which are for the first time given in this paper) and the reflection matrices for the sine-Gordon breathers, which were given by Ghoshal in [17]. The second step will be to add a factor $R_{BF}(\theta)$ that describes the relative amplitudes for bosons and fermions when scattering off the boundary. The factor $R_{BF}(\theta)$ is subject to a non-trivial BYBE. Our strategy will be to derive $R_{BF}(\theta)$ by imposing supersymmetry and to check that the resulting expression indeed satisfies the BYBE.

In the presence of bound states (bulk or boundary), there are consistency requirements that are not automatically satisfied by the factorized expression (3.5). These will be discussed below.

Boundary Supersymmetry

In the presence of a boundary, only a specific linear combination of the left and right supersymmetries can be preserved. This is of course familiar from open string theory, see also [14]. The invariant combination should satisfy the “commutation” relation

$$Q(\theta)R(\theta) = R(\theta)Q(-\theta) , \quad (3.6)$$

where $Q(\theta) = aQ_+(\theta) + bQ_-(\theta)$, and a and b are constants. It is easy to see that the only solutions to these equation are simply

$$Q^{(\pm)}(\theta) = Q_+(\theta) \mp Q_-(\theta) , \quad (3.7)$$

together with

$$R_{BF}^{(\pm)}(\theta) = Z^{(\pm)}(\theta) \begin{pmatrix} \cosh(\frac{\theta}{2} \pm i\frac{\pi}{4}) & e^{i\frac{\pi}{4}}Y(\theta) \\ e^{-i\frac{\pi}{4}}Y(\theta) & \cosh(\frac{\theta}{2} \mp i\frac{\pi}{4}) \end{pmatrix} . \quad (3.8)$$

These are thus the most general reflection matrices compatible with our realization of $N = 1$ supersymmetry (which assumed the absence of topological charges). If we impose now that we have a boundary with no structure we should set $Y(\theta) = 0$, since $Y(\theta)$ is (up to a phase)

the amplitude of a scattering off the boundary that changes the fermion number. The result is then

$$R_{BF}^{(\pm)}(\theta) = Z^{(\pm)}(\theta) \begin{pmatrix} \cosh(\frac{\theta}{2} \pm i\frac{\pi}{4}) & 0 \\ 0 & \cosh(\frac{\theta}{2} \mp i\frac{\pi}{4}) \end{pmatrix} . \quad (3.9)$$

It is surprising that we have been able to determine the form of the possible supersymmetry charges and the structure of the reflection matrix in one go. The result is that the amplitudes for a particle and its superpartner scattering off the boundary are related in a universal way (independent of masses) by

$$\frac{R_b^{(\pm)}(\theta)}{R_f^{(\pm)}(\theta)} = \frac{\cosh(\frac{\theta}{2} \pm i\frac{\pi}{4})}{\cosh(\frac{\theta}{2} \mp i\frac{\pi}{4})} . \quad (3.10)$$

It is important to stress that this result assumes that we started with an integrable supersymmetric model and introduced a boundary that preserves *both* integrability and supersymmetry. We just learned that these conditions are very restrictive and in fact leave no free parameters (except for a choice of sign) in the choice of boundary conditions. A similar conclusion has been reached in a purely classical analysis of the susy sine-Gordon theory [1]. In section VI we shall propose a precise connection between our results and the work of [1].

In order to fix the prefactor $Z^{(\pm)}(\theta)$ we have to use the unitarity and boundary crossing-unitarity conditions. They lead to

$$\begin{aligned} Z_j^{(\pm)}(\theta) Z_j^{(\pm)}(-\theta) &= \frac{2}{\cosh(\theta)} , \\ \frac{Z_j^{(\pm)}(i\frac{\pi}{2} - \theta)}{Z_j^{(\pm)}(i\frac{\pi}{2} + \theta)} &= \mp S_{b_j b_j}^{b_j b_j}(2\theta) + i f^{(\pm)}(\theta) S_{f_j f_j}^{b_j b_j}(2\theta) , \end{aligned} \quad (3.11)$$

where $f^{(+)}(\theta) = \coth(\frac{\theta}{2})$ and $f^{(-)}(\theta) = \tanh(\frac{\theta}{2})$, depending on which sign we pick in $R_{BF}^{(\pm)}$. The index j in the notation $Z_j^{(\pm)}(\theta)$ expresses the dependence of these functions on the mass m_j of the reflecting particle.

In section V we shall present explicit solutions to the equations (3.11). Such solutions are unique up to multiplicative factors $\phi(\theta)$ satisfying

$$\phi(\theta)\phi(-\theta) = 1, \quad \frac{\phi(i\frac{\pi}{2} + \theta)}{\phi(i\frac{\pi}{2} - \theta)} = 1. \quad (3.12)$$

Such factors are the boundary analogues of the familiar CDD factors.

The Free Case

We can apply the preceding discussion to an extremely simple, but nonetheless useful, example: the free supersymmetric theory with one boson and one fermion. In this case we have $S_B(\theta) = 1$ and the non-vanishing entries of $S_{BF}(\theta)$ are given by $S_{bb}^{bb} = 1$, $S_{ff}^{ff} = -1$, and $S_{bf}^{bf} = S_{fb}^{fb} = 1$. Clearly, there are no bound states that we should worry about. Assuming the trivial amplitude $R_b(\theta) = 1$, we find

$$R^{(\pm)}(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\cosh(\frac{\theta}{2} \mp i\frac{\pi}{4})}{\cosh(\frac{\theta}{2} \pm i\frac{\pi}{4})} \end{pmatrix}. \quad (3.13)$$

Comparing with the result by Ghoshal and Zamolodchikov [13] for the free Majorana fermion, we see that $R^{(+)}(\theta)$ corresponds to the free boundary condition $(\psi - \bar{\psi} = 0)$ and $R^{(-)}(\theta)$ to the fixed boundary condition $(\psi + \bar{\psi} = 0)$.

Boundary Bootstrap and Boundary Bound States

The presence of poles (with imaginary part between 0 and $\pi/2$) in a boundary reflection matrix signals a resonance, where bulk particles can ‘merge’ with the boundary state.

A resonance at rapidity $i\frac{\pi}{2}$, i.e., at zero energy, indicates that the (unique) boundary state $|0\rangle_B$ has a non-zero coupling g^c with zero-energy bulk particles of type c . If there is a non-zero bulk coupling f_{ab}^c with $m_a = m_b$, one expects a pole of $R_a^b(\theta)$ at $\theta = i\frac{\bar{u}_{ab}^c}{2}$ with

$$R_a^b(\theta) \sim -\frac{i}{2} \frac{f_{ab}^c g^c}{\theta - i\frac{\bar{u}_{ab}^c}{2}}, \quad (3.14)$$

as $\theta \rightarrow i\frac{\bar{u}_{ab}^c}{2}$.

In the presence of $N = 1$ supersymmetry, one would in principle expect that the bound state couplings g^i branch into non-zero g^{b_i} and g^{f_i} . However, in view of (3.14), a non-zero value for g^{f_i} would imply singularities in amplitudes that describe reflection processes that violate fermion number. Since we assumed that such processes do not occur, we conclude

that all fermionic couplings g^{f_i} should vanish. This then implies that the ratio R_f/R_b should have a zero at $\theta = i\frac{\pi}{2}$, which is the case for $R_{BF}^{(-)}$ but not for $R_{BF}^{(+)}$. We conclude that in the situation with non-vanishing boundary couplings g^i , the supersymmetrization can only be done with the factor $R_{BF}^{(-)}$.

If we now assume a non-vanishing coupling among the particles (b_i, b_i, b_j) , we derive that

$$R_{b_i}(\theta) \sim -\frac{i}{2} \frac{f_{b_i b_i}^{b_j} g^{b_j}}{\theta - i\frac{\bar{w}_{ii}^j}{2}}, \quad R_{f_i}(\theta) \sim -\frac{i}{2} \frac{f_{f_i f_i}^{b_j} g^{b_j}}{\theta - i\frac{\bar{w}_{ii}^j}{2}}, \quad (3.15)$$

and taking the ratio we obtain

$$\frac{R_{b_i}(i\frac{\bar{w}_{ii}^j}{2})}{R_{f_i}(i\frac{\bar{w}_{ii}^j}{2})} = \frac{f_{b_i b_i}^{b_j}}{f_{f_i f_i}^{b_j}}. \quad (3.16)$$

The RHS of this equation is a feature of the bulk scattering theory, and it depends solely on the masses m_i and m_j (see (2.14)). The LHS (which does not depend on the normalization $Z^{(\pm)}(\theta)$), is precisely the universal ratio (3.10) (based on $R_{BF}^{(-)}$) of the boson and fermion reflection amplitudes. Thus, while the value of the RHS is a consequence of the bulk supersymmetry, the LHS has been dictated by boundary supersymmetry. The fact that these ratios do indeed agree (as is easily worked out) is therefore a nice confirmation of the consistency of our description.

If the boundary reflection amplitudes have resonances that can not be taken into account by the above, this may signal the presence of ‘boundary bound states’. These are stable configurations in which the boundary can exist and which can be excited by incident bulk particles. A boundary bound state $|\alpha\rangle_B$ of energy e_α can be excited by an incident particle of mass m and rapidity $\theta = iv_{0a}^\alpha$, provided

$$e_0 + m_a \cos(v_{0a}^\alpha) = e_\alpha. \quad (3.17)$$

The behavior of $R_a^b(\theta)$ is then given by

$$R_a^b(\theta) \sim \frac{i}{2} \frac{g_{a0}^\alpha g_\alpha^{b0}}{\theta - iv_{0a}^\alpha}, \quad (3.18)$$

as $\theta \rightarrow iv_{0a}^\alpha$. In the presence of supersymmetry, a boundary coupling g_{a0}^α will branch into a coupling g_{b0}^β for the bosons and a coupling g_{f0}^φ for the fermions. In other words, there are

boundary states $|\beta\rangle_B$ and $|\varphi\rangle_B$, which form a multiplet under $N = 1$ boundary supersymmetry. The relative strength of g_{b0}^β and g_{f0}^φ can for example be derived from the observation that the residue of $R_b(\theta)$ ($R_f(\theta)$) at the pole $\theta \rightarrow iv_{b0}^\beta$ is proportional to $(g_{b0}^\beta)^2$ ($(g_{f0}^\varphi)^2$). This implies that

$$\left(\frac{g_{b0}^\beta}{g_{f0}^\varphi}\right)_{(\pm)}^2 = \frac{\cosh(\frac{iv}{2} \pm i\frac{\pi}{4})}{\cosh(\frac{iv}{2} \mp i\frac{\pi}{4})} \quad (3.19)$$

with $v = v_{b0}^\beta$.

We can formally write the excited boundary states as

$$\lim_{\epsilon \rightarrow 0} \epsilon b(iv + \epsilon) |0\rangle_B = g_{b0}^\beta |\beta\rangle_B, \quad \lim_{\epsilon \rightarrow 0} \epsilon f(iv + \epsilon) |0\rangle_B = g_{f0}^\varphi |\varphi\rangle_B. \quad (3.20)$$

Using (3.19) we can work out the action of boundary supersymmetry on these states, with the result

$$\begin{aligned} Q^{(+)}|\beta\rangle_B &= \sqrt{2m \cos v} |\varphi\rangle_B, & Q^{(+)}|\varphi\rangle_B &= \sqrt{2m \cos v} |\beta\rangle_B, \\ Q^{(-)}|\beta\rangle_B &= i\sqrt{2m \cos v} |\varphi\rangle_B, & Q^{(-)}|\varphi\rangle_B &= -i\sqrt{2m \cos v} |\beta\rangle_B. \end{aligned} \quad (3.21)$$

We see that $[Q^{(\pm)}]^2 = 2m \cos v = 2(e_\beta - e_0)$ as it should be.

Focusing on the case with $R_{BF}^{(-)}$, we can imagine starting from the situation with $v = \pi/2$ and then moving v down along the imaginary axis. In the supersymmetric theory, this process represents a kind of supersymmetry breaking at the boundary: at $v = \pi/2$ the boundary state has zero energy and it is annihilated by supersymmetry, while for $v < \pi/2$ the energy is positive and supersymmetry no longer annihilates the state.

In the presence of boundary bound states, one may consider more general reflection amplitudes, such as $R_{a\alpha}^{b\beta}$, which describes a process where a particle $a(\theta)$ reflects into $b(-\theta)$ while the boundary makes a transition from $|\alpha\rangle_B$ to $|\beta\rangle_B$. Clearly, our formalism can be extended to amplitudes of this type, but we shall not do so in this paper.

As we will see later, not all bosonic reflection matrices can be supersymmetrized, even though we have a general form for $R_{BF}(\theta)$. This can probably be understood through the analysis of residue conditions for more general reflection amplitudes, such as $R_{a\alpha}^{b\beta}$, which

involve boundary bound states [14]. In the bulk theories, such an analysis [5] has led to the condition (2.15), which shows that many bosonic theories simply can not be supersymmetrized by adding a factor S_{BF} to their bosonic S -matrix S_B . In a similar way, we expect that the product form $R = R_B \otimes R_{BF}$ for a supersymmetric reflection matrix will only be possible in theories with a specific and restricted set of boundary bound states.

IV. BOSONIC REFLECTION MATRICES

As we have discussed earlier, we need to find the reflection matrices for the bosonic sectors of the models we are studying. The reflection matrices for the FKM series (which are minimal reductions of the $A_{2n}^{(2)}$ affine Toda theories) have been studied for a few special cases only [13,15]. Here we shall give two solutions for general $n = 1, 2, \dots$. The reflection matrices for bound states of the boundary sine-Gordon theory were studied by Ghoshal in [17]; for completeness we quote his results below. We shall also comment on the relation between the reflection matrices for sine-Gordon breathers and those for the FKM models.

The FKM Series

In this subsection we derive two possible reflection matrices for each of the bosonic FKM models.

We begin our derivation by briefly reviewing the work of Fring and Köberle [15,16] on the reflection matrices for the affine Toda field theories based on the untwisted affine algebras $A_l^{(1)}$. These are massive, integrable QFT's, with particles of masses m_j , $j = 1, 2, \dots, l$, given by the formula (2.17) with $\beta = l + 1$. Starting from a “block” structure for the S -matrix, Fring and Köberle were able to conjecture the following form of the reflection matrices

$$R_{A_l^{(1)}}^j(\theta) = \prod_{i=1}^{\mu(j)} \mathcal{W}_{l-2\mu(j)+2\nu(i)}(\theta) , \quad (4.1)$$

with $\mu(i)$ given by

$$\mu(i) = \begin{cases} i & \text{for } i \leq [\frac{h}{2}] \\ h - i & \text{for } i > [\frac{h}{2}] \end{cases} , \quad (4.2)$$

and $\nu(i)$ by

$$\nu(i) = \begin{cases} i & \text{for } i \text{ odd} \\ i + h & \text{for } i \text{ even} \end{cases}, \quad (4.3)$$

where h is the Coxeter number associated to the affine Lie algebra $A_l^{(1)}$, $h = l+1$. The precise expression for the building block $\mathcal{W}_x(\theta)$, which may be found in [16], of course depends on the value of the coupling constant of the affine Toda theory. The expressions (4.1) are free of poles in the physical strip.

We shall take this result as a starting point for deriving the reflection matrices for FKM series of minimal models. To get there, we need to make two steps. The first step is to reduce the result for the $A_l^{(1)}$ Toda series to parameter-free ‘minimal’ reflection matrices. These then describe the ‘minimal’ boundary scattering of a set of l particles, whose bulk S -matrix was first given in [18]. This minimal scattering theory describes the IR behavior of an integrable perturbation of a CFT of central charge $c_l = \frac{2l}{l+3}$. In the above formulas, the reduction to the minimal case can be done by replacing the $\mathcal{W}_x(\theta)$ used in [16] by the following parameter-free expression

$$W_x(\theta) = \frac{\sinh \frac{1}{2}(\theta + i\frac{\pi(1-x-h)}{2h}) \sinh \frac{1}{2}(\theta - i\frac{\pi(1+x+h)}{2h})}{\sinh \frac{1}{2}(\theta - i\frac{\pi(1-x-h)}{2h}) \sinh \frac{1}{2}(\theta + i\frac{\pi(1+x+h)}{2h})}. \quad (4.4)$$

The second step is to apply a procedure called ‘folding in half’ to the minimal reduction of the $A_l^{(1)}$ Toda result with $l = 2n$. It is well-known that the bulk S -matrices for the minimal reductions of the $A_{2n}^{(1)}$ and $A_{2n}^{(2)}$ affine Toda theories satisfy the relation

$$S_{ij}^{A_{2n}^{(2)}}(\theta) = S_{ij}^{A_{2n}^{(1)}}(\theta) S_{i\bar{j}}^{A_{2n}^{(1)}}(\theta) \quad (4.5)$$

with $i, j = 1, 2, \dots, n$ and $\bar{j} = (h - j)$. This folding relation has interesting consequences at the level of the TBA based on these S -matrices. In particular, it follows that the effective central charge of the minimal $A_{2n}^{(2)}$ theory is precisely half that of the theory based on $A_{2n}^{(1)}$. The folding relation of the bulk S -matrices suggests a similar structure for the reflection matrices, where we expect that every factor $W_x(\theta)$ will be multiplied by a factor $W_{h-x}(\theta)$.

This then is up to possible shifts of the indices x by the amount $2h = 4n + 2$. These we have determined ‘by hand’, by insisting on a consistent pole structure and on the boundary bootstrap conditions.

We claim that the following two series of reflection matrices are consistent solutions for the FKM minimal theories

$$\begin{aligned} R_{(1)}^j(\theta) &= \prod_{i=1}^j \left[W_{2n-2j+2\nu(i)}(\theta) W_{-4n+2j-1-2i}(\theta) \right] , \\ R_{(2)}^j(\theta) &= \prod_{i=1}^j \left[W_{2n-2j+2\nu(i)}(\theta) W_{2j+1-2i}(\theta) \right] . \end{aligned} \quad (4.6)$$

We can check this conjecture against a few known cases. Ghoshal and Zamolodchikov studied the Yang-Lee model (the $n = 1$ model in the FKM series) in [13]. For $n = 1$ (4.6) gives

$$R_{(1)}^1(\theta) = W_2(\theta)W_{-5}(\theta) , \quad R_{(2)}^1(\theta) = W_2(\theta)W_1(\theta) , \quad (4.7)$$

in agreement with the result of [13]⁴. For $n = 2$ the result is⁵

$$\begin{aligned} R_{(1)}^1(\theta) &= W_4(\theta)W_{-9}(\theta) , \quad R_{(1)}^2(\theta) = [W_2(\theta)W_{-7}(\theta)] [W_{-6}(\theta)W_{-9}(\theta)] , \\ R_{(2)}^1(\theta) &= W_4(\theta)W_1(\theta) , \quad R_{(2)}^2(\theta) = [W_2(\theta)W_3(\theta)] [W_{-6}(\theta)W_1(\theta)] . \end{aligned} \quad (4.8)$$

In general, the solution $R_{(2)}^j(\theta)$ is obtained from $R_{(1)}^j(\theta)$ by the replacement $W_x(\theta) \rightarrow W_{x+2h}(\theta)$ for all odd x . It can easily be checked that this replacement corresponds to a multiplication by a CDD factor that is consistent with the boundary bootstrap equations.

The reflection amplitude $R_{(1)}^1(\theta)$ for the lightest particle has ‘physical’ poles at $\theta = i\frac{(2n-1)\pi}{2(2n+1)}$, and $\theta = i\frac{\pi}{2}$. The first of these is at $\theta = i\frac{\bar{u}_{11}^2}{2}$ and corresponds to a nonzero boundary coupling g^2 . As we have seen, the pole at $\theta = i\frac{\pi}{2}$ indicates a non-zero value of g^1 . The reflection amplitude $R_{(2)}^1(\theta)$ is free of poles in the physical strip.

⁴There is a sign mistake in the equation (3.51) of [13].

⁵ $\mathcal{W}_4(\theta)$ should be replaced by $\mathcal{W}_{-6}(\theta)$ for the $A_4^{(2)}$ in Fring and Köberle’s [15] equation (5.81).

The sine-Gordon Bound States

The boundary sine-Gordon model was studied in [13,17], where these authors found the reflection matrices for solitons and breathers. We shall need the latter ones. Most importantly, it was found that the sine-Gordon reflection matrices have two free parameters, called η and ϑ . These parameters are believed to correspond to free parameters M and ϕ_0 in the most general boundary action that preserves the integrability of the sine-Gordon theory.

Following Ghoshal [17], we write the reflection matrix for the j -th breather as $R_{sG}^j(\theta|\eta, \vartheta) = R_0^j(\theta) R_1^j(\theta)$, where

$$R_0^j(\theta) = (-1)^{j+1} \frac{\cosh(\frac{\theta}{2} + i\frac{j\pi}{4\lambda}) \cosh(\frac{\theta}{2} - i\frac{\pi}{4} - i\frac{j\pi}{4\lambda}) \sinh(\frac{\theta}{2} + i\frac{\pi}{4})}{\cosh(\frac{\theta}{2} - i\frac{j\pi}{4\lambda}) \cosh(\frac{\theta}{2} + i\frac{\pi}{4} + i\frac{j\pi}{4\lambda}) \sinh(\frac{\theta}{2} - i\frac{\pi}{4})} \times \prod_{k=1}^{j-1} \frac{\sinh(\theta + i\frac{k\pi}{2\lambda}) \cosh^2(\frac{\theta}{2} - i\frac{\pi}{4} - i\frac{k\pi}{4\lambda})}{\sinh(\theta - i\frac{k\pi}{2\lambda}) \cosh^2(\frac{\theta}{2} + i\frac{\pi}{4} + i\frac{k\pi}{4\lambda})} . \quad (4.9)$$

The coupling parameters (η, ϑ) are contained in $R_1^j(\theta)$, which can be written as

$$R_1^j(\theta) = S^j(\eta, \theta) S^j(i\vartheta, \theta) , \quad (4.10)$$

where $S^j(x, \theta)$ depends whether j is even or odd. For $j = 2k$, $k = 1, 2, \dots < \frac{\lambda}{2}$, we have

$$S^{2k}(x, \theta) = \prod_{l=1}^k \frac{\sinh(\theta) - i \cos(\frac{x}{\lambda} - (l - \frac{1}{2})\frac{\pi}{\lambda}) \sinh(\theta) - i \cos(\frac{x}{\lambda} + (l - \frac{1}{2})\frac{\pi}{\lambda})}{\sinh(\theta) + i \cos(\frac{x}{\lambda} - (l - \frac{1}{2})\frac{\pi}{\lambda}) \sinh(\theta) + i \cos(\frac{x}{\lambda} + (l - \frac{1}{2})\frac{\pi}{\lambda})} . \quad (4.11)$$

For $j = 2k - 1$, $k = 1, 2, \dots < \frac{\lambda+1}{2}$, we have

$$S^{2k-1}(x, \theta) = \frac{i \cos(\frac{x}{\lambda}) - \sinh(\theta)}{i \cos(\frac{x}{\lambda}) + \sinh(\theta)} \prod_{l=1}^{k-1} \frac{\sinh(\theta) - i \cos(\frac{x}{\lambda} - \frac{l\pi}{\lambda}) \sinh(\theta) - i \cos(\frac{x}{\lambda} + \frac{l\pi}{\lambda})}{\sinh(\theta) + i \cos(\frac{x}{\lambda} - \frac{l\pi}{\lambda}) \sinh(\theta) + i \cos(\frac{x}{\lambda} + \frac{l\pi}{\lambda})} . \quad (4.12)$$

V. REFLECTION MATRICES FOR THE SUPERSYMMETRIC FKM MODELS

In the previous section we obtained the reflection matrices for the FKM models. In order to write down the complete reflection matrix for the supersymmetric generalizations we just have to “attach” the $R_{BF}(\theta)$ as given in (3.9), giving

$$R^{[j]}(\theta) = R_{FKM}^j(\theta) Z_j^{(\pm)}(\theta) \begin{pmatrix} \cosh(\frac{\theta}{2} \pm i\frac{\pi}{4}) & 0 \\ 0 & \cosh(\frac{\theta}{2} \mp i\frac{\pi}{4}) \end{pmatrix} , \quad (5.1)$$

where R_{FKM}^j stands for either $R_{(1)}^j$ or $R_{(2)}^j$ of (4.6). Before continuing we should point out that the choice of (1) or (2) in the bosonic factor is correlated with the choice of (+) or (-) in the supersymmetric factor. We already mentioned that the solution $R_{(1)}$ (which is the one with non-vanishing boundary couplings) should be combined with $R_{BF}^{(-)}$, and it is then natural to associate $R_{(2)}$ with $R_{BF}^{(+)}$. We thus propose the following complete reflection matrices

$$R_{(1)}^{[j]} = R_{(1)}^j R_{BF}^{(-)}, \quad R_{(2)}^{[j]} = R_{(2)}^j R_{BF}^{(+)} . \quad (5.2)$$

Until now we have not specified the normalization factors $Z_j^{(\pm)}(\theta)$ of the supersymmetric reflection matrices $R_{BF}^{(\pm)}$. Up to CDD factors, these are fixed by the equations (3.11) that express unitarity and boundary crossing unitarity. We shall now present explicit expressions for these factors.

The right hand side of the second equation of (3.11) depends on the mass m_j of the reflecting particle via the combination βj , where $\beta = 1/(2n+1)$ for the susy FKM series, $\beta = 1/(2\lambda)$ for susy sine-Gordon and $\beta \rightarrow 0$ for the free theory.

Let us consider separately the two possibilities ‘(-)’ and ‘(+)’. In the first case it is convenient to write $Z_j^{(-)}(\theta)$ as

$$Z_j^{(-)}(\theta) = \frac{1}{\cosh(\frac{\theta}{2} - i\frac{\pi}{4})} \tilde{Z}_j^{(-)}(\theta) , \quad (5.3)$$

with $\tilde{Z}_j^{(-)}(\theta)\tilde{Z}_j^{(-)}(-\theta) = 1$. The second equation of (3.11) then leads to

$$\frac{\tilde{Z}_j^{(-)}(i\frac{\pi}{2} - \theta)}{\tilde{Z}_j^{(-)}(i\frac{\pi}{2} + \theta)} = \exp \left(i \int_0^\infty \frac{dt}{t} \frac{\sinh(\beta j t) \sinh((1 - \beta j)t)}{\cosh^2(\frac{t}{2}) \cosh(t)} \sin(\frac{2t\theta}{\pi}) \right) . \quad (5.4)$$

By elementary methods we solve for $\tilde{Z}_j^{(-)}(\theta)$ and obtain

$$Z_j^{(-)}(\theta) = \frac{1}{\cosh(\frac{\theta}{2} - i\frac{\pi}{4})} \exp \left(-\frac{i}{2} \int_0^\infty \frac{dt}{t} \frac{\sinh(\beta j t) \sinh((1 - \beta j)t)}{\cosh^2(\frac{t}{2}) \cosh^2(t)} \sin(\frac{2t\theta}{\pi}) \right) . \quad (5.5)$$

In an analogous fashion we make the following Ansatz for $Z_j^{(+)}(\theta)$

$$Z_j^{(+)}(\theta) = \frac{1}{\cosh(\frac{\theta}{2} + i\frac{\pi}{4})} \tilde{Z}_j^{(+)}(\theta) , \quad (5.6)$$

with $\tilde{Z}_j^{(+)}(\theta)\tilde{Z}_j^{(+)}(-\theta) = 1$. This leads to

$$\frac{\tilde{Z}_j^{(+)}(i\frac{\pi}{2} - \theta)}{\tilde{Z}_j^{(+)}(i\frac{\pi}{2} + \theta)} = \frac{\sinh(\theta) - i \sin(\beta j \pi)}{\sinh(\theta) + i \sin(\beta j \pi)} \exp \left(i \int_0^\infty \frac{dt}{t} \frac{\sinh(\beta j t) \sinh((1 - \beta j)t)}{\cosh^2(\frac{t}{2}) \cosh(t)} \sin(\frac{2t\theta}{\pi}) \right) . \quad (5.7)$$

We can rewrite the prefactor by using that for $\theta \neq 0$ and real⁶

$$\frac{\sinh(\theta) - i \sin(\alpha \pi)}{\sinh(\theta) + i \sin(\alpha \pi)} = \exp \left(4i \int_0^\infty \frac{dt}{t} \frac{\cosh(\frac{\alpha}{2}t) \cosh(\frac{1-\alpha}{2}t)}{\cosh^2(\frac{t}{2})} \sin(\frac{t\theta}{\pi}) \right) , \quad (5.8)$$

and by the same elementary methods we obtain

$$\begin{aligned} Z_j^{(+)}(\theta) &= \frac{1}{\cosh(\frac{\theta}{2} + i\frac{\pi}{4})} \exp \left(-2i \int_0^\infty \frac{dt}{t} \frac{\cosh(\frac{1}{2}\beta j t) \cosh(\frac{1}{2}(1 - \beta j)t)}{\cosh^2(\frac{t}{2})} \sin(\frac{t\theta}{\pi}) \right) \\ &\quad \times \exp \left(-\frac{i}{2} \int_0^\infty \frac{dt}{t} \frac{\sinh(\beta j t) \sinh((1 - \beta j)t)}{\cosh^2(\frac{t}{2}) \cosh^2(t)} \sin(\frac{2t\theta}{\pi}) \right) . \end{aligned} \quad (5.9)$$

Note that in the free limit, $\beta \rightarrow 0$, these Z -factors reproduce the ones that we used in (3.13).

VI. REFLECTION MATRICES FOR BOUND STATES OF SUSY SINE-GORDON

In ref. [1], it was found that there are two special choices for boundary conditions on the *classical* susy sine-Gordon theory such that the resulting theory is both integrable and supersymmetric. Expressed in the variables of the action (2.20), these conditions are

$$\mathcal{BC}^\pm : \quad \partial_x \phi \pm \frac{2m}{\beta_{ssG}} \sin(\frac{\beta_{ssG} \phi}{2}) = 0 , \quad \psi \mp \bar{\psi} = 0 \quad (6.1)$$

at $x = 0$. In the notation of [13], the bosonic part of these conditions corresponds to $M = M_\pm = \pm \frac{4m}{\beta^2}$, $\phi_0 = 0$. We shall refer to these boundary conditions as \mathcal{BC}^\pm .

We conjecture that at those two special points, the reflection matrix for the bound state multiplets (b_j, f_j) , $j = 1, 2, \dots < \lambda$ will be of the form $R_B \otimes R_{BF}$, where R_B is a

⁶This will be sufficient for the purpose of boundary TBA calculations.

reflection matrix for the breathers of the sine-Gordon theory, see (4.9) – (4.12) and R_{BF} is the supersymmetric reflection matrix of section III.

Two questions arise: what are the values of η and ϑ for these two special points, and how can we match the choice of sign in the boundary conditions \mathcal{BC}^\pm with the choice of sign in the reflection matrix $R_{BF}^{(\pm)}$?

Part of the answer to the first question can be found in [13], where it is shown that the $\phi_0 = 0$ boundary potentials correspond to a condition $\xi = 0$ that is satisfied when $\vartheta = 0$. If we now look at the Ghoshal reflection matrix $R_{sG}^1(\theta|\eta, \vartheta = 0)$ (for the first breather, $j = 1$) for generic η , we see that it has zeros at $\theta = i\frac{\pi}{2}, i\pi\frac{2\lambda-1}{2\lambda}, i\pi\frac{1-\lambda}{2\lambda}$. We can then look for special values of η that are such that one of these zero's gets canceled. This happens when $\eta = \lambda\pi, \frac{\pi}{2}(\lambda + 1), \frac{\pi}{2}$, resp. The point $\eta = \frac{\pi}{2}(\lambda + 1)$ has been identified with the value $M = 0$ (free boundary conditions in the bosonic sector), see [13].

At the special values $\lambda = \frac{2n+1}{2}$ the minimal choices $\eta = \lambda\pi$ and $\eta = \frac{\pi}{2}$ precisely give the two minimal reflection matrices $R_{(1)}$ and $R_{(2)}$ for the n -cpt FKM minimal models as given in our section IV.

Combining these observations, it is natural to conjecture that the two points that correspond to an integrable, supersymmetric boundary potential are precisely the two points $\eta = \lambda\pi, \frac{\pi}{2}$ where the reflection matrix is ‘minimal’ with non-zero M .

What remains to be specified is which of the two factors $R_{BF}^{(\pm)}(\theta)$ should be used in the two cases at hand, and also what the correspondence with the boundary conditions \mathcal{BC}^\pm is. The minimal solution with $\eta = \pi\lambda$, called $R_{(1)}$ in the context of the FKM minimal models, has non-vanishing boundary couplings and should thus be combined with $R_{BF}^{(-)}$ (see section III). This leaves $R_{BF}^{(+)}$ to be combined with the solution at $\eta = \frac{\pi}{2}$

Let us now recall that the ratio R_f/R_b of the amplitudes in the factor $R_{BF}^{(\pm)}$ precisely corresponds to the reflection amplitude for a Majorana fermion, with free boundary condition, $\psi - \bar{\psi} = 0$, in the case of $R_{BF}^{(+)}$ and fixed boundary condition, $\psi + \bar{\psi} = 0$, in the case of $R_{BF}^{(-)}$ (see section III). The fermionic components of the boundary conditions \mathcal{BC}^\pm are precisely of this same form, and this suggests that we link \mathcal{BC}^+ to $R_{BF}^{(+)}$ and \mathcal{BC}^- to $R_{BF}^{(-)}$. This iden-

tification gets further confirmed by the correspondence between the our supercharges $Q^{(\pm)}$ and the susy transformations given in [1]. We therefore expect that the complete result for the reflection matrices of the breather multiplets of susy sine-Gordon are

$$\begin{aligned} R_{sG}(\theta|\eta = \frac{\pi}{2}, \vartheta = 0) R_{BF}^{(+)}(\theta) & \quad \text{for} \quad \mathcal{BC}^+ \\ R_{sG}(\theta|\eta = \lambda\pi, \vartheta = 0) R_{BF}^{(-)}(\theta) & \quad \text{for} \quad \mathcal{BC}^- . \end{aligned} \quad (6.2)$$

At the special values $\lambda = \frac{2n+1}{2}$ these reflection matrices reduce to the two supersymmetric reflection matrices that we propose for the susy FKM models.

VII. CONCLUSIONS

We have proposed exact reflections matrices for a number of integrable $N = 1$ supersymmetric theories. The examples studied are the supersymmetric sine-Gordon theory and a series of minimal models (called susy FKM) that arise as perturbations of superconformal field theories. These theories all have exact (bulk) S -matrices that enjoy the factorization as written in (2.8).

The main result of this paper is the structure of the reflection matrix (3.9), which was obtained directly from the following two assumptions: (i) both integrability and supersymmetry are preserved after the introduction of the boundary and (ii) the boundary has no structure, that is, it does not allow scattering processes that violate fermion number.

In order to write down the complete reflection matrix for the models studied in this paper we started from the reflection matrices for their bosonic reductions. In the case of sine-Gordon model, these were given by Ghoshal in [17]. For the susy FKM models such reflection matrices are for the first time given in this paper. We derived them by exploiting a connection with the $A_{2n}^{(2)}$ twisted affine Toda theories.

In [1] Inami, Odake and Zhang proposed two possible boundary conditions that preserve supersymmetry and integrability upon the introduction of a boundary in the susy sine-Gordon model. Their result was based on a classical analysis of the constraints imposed by

the first non-trivial conserved charge. We conjectured a connection between the reflection matrices found in this paper and the boundary conditions proposed in [1].

There are a number of issues that deserve further attention. It would be interesting to understand better the origin of the two distinct reflection matrices $R_{(1)}$ and $R_{(2)}$ for the susy FKM models, for example by first studying boundary fields in the unperturbed superconformal field theory. Another problem that deserves consideration would be to perform a TBA analysis, generalizing the methods used in [9,6] from the bulk system to the boundary system, and to investigate the connections with $N = 2$ theories as was done in [9] for the bulk theory. One may also consider more general (possibly non-supersymmetric) solutions to the boundary Yang-Baxter equations for the S -matrices studied in this paper.

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